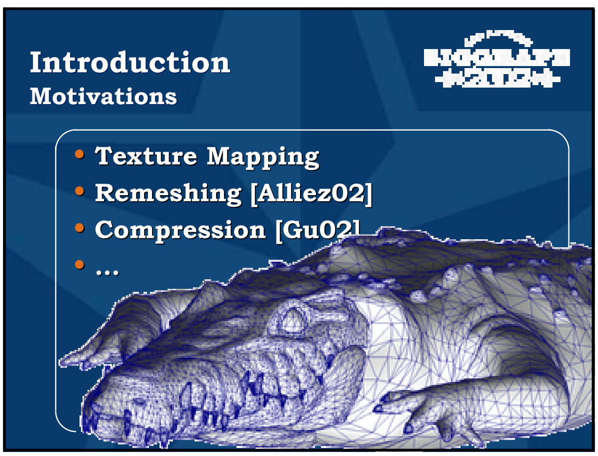
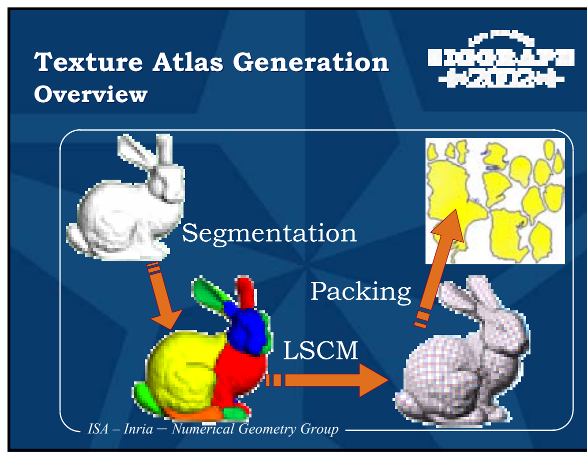
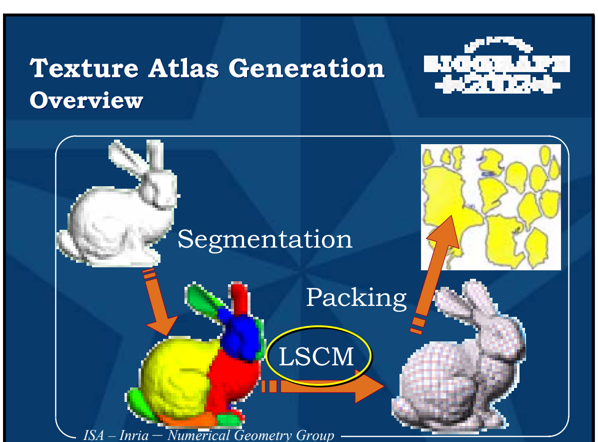
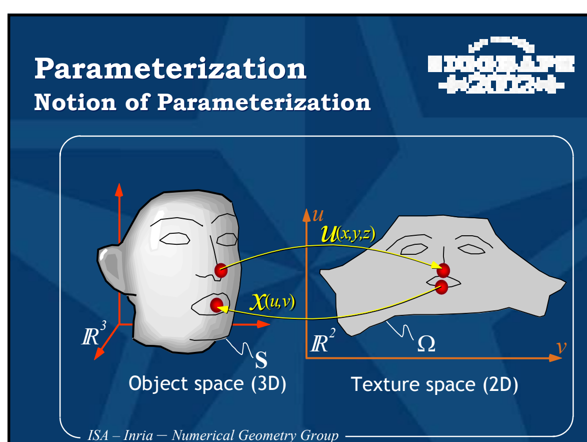

SAN ANTONIO
SIGGRAPH
2002


SIGGRAPH 2002
Least Squares Conformal Maps
Bruno Lévy
Sylvain Petitjean
Nicolas Ray
Jérome Maillot

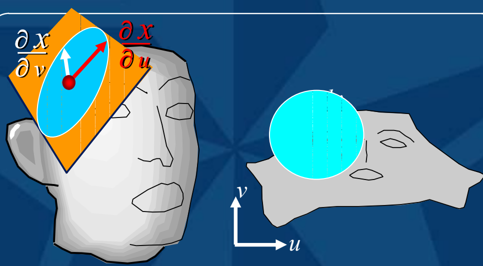

Introduction
Motivations
SIGGRAPH 2002
• Texture Mapping
• Remeshing [Alliez02]
• Compression [Gu02]
• …


Texture Atlas Generation
Overview
SIGGRAPH 2002
Segmentation
Packing
LSCM
ISA – Inria — Numerical Geometry Group


Texture Atlas Generation
Overview
SIGGRAPH 2002
Segmentation
Packing
LSCM
ISA – Inria — Numerical Geometry Group

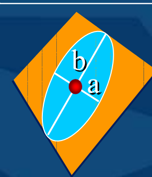

Parameterization
Notion of Parameterization
SIGGRAPH 2002
u
U(x,y,z)
X(u,v)
R3
R2
S
Ω
v
Object space (3D)
Texture space (2D)
ISA – Inria — Numerical Geometry Group

Parameterization
Anisotropy - partial derivatives



Object space (3D) Texture space (2D)

Parameterization
Anisotropy - 1st fundamental form

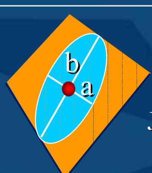


$$G = \begin{bmatrix} \frac{\partial x}{\partial u}^2 & \frac{\partial x}{\partial u} . \frac{\partial x}{\partial v} \\ \frac{\partial x}{\partial u} . \frac{\partial x}{\partial v} & \frac{\partial x}{\partial v}^2 \end{bmatrix}$$

$$\| x(W) \|^2 = W^t.G.W$$

$a = \sqrt{\lambda_1}$; $b = \sqrt{\lambda_2}$ *(eigen values of G)*

Parameterization
Anisotropy – Jacobian

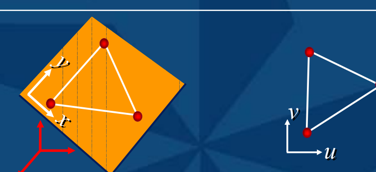


$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = U \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} V^t$$

Singular values decomposition (SVD) of J

Rem: $G = J^t.J \Longrightarrow a = \sqrt{\lambda_1}$; $b = \sqrt{\lambda_2}$

Parameterization
Piecewise Linear Parameterization



From (x,y,z) to (u,v)

$u(x,y) = a_1.x + b_1.y + c_1$

$v(x,y) = a_2.x + b_2.y + c_2$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_2\text{-}y_3 & y_3\text{-}y_1 & y_1\text{-}y_2 \\ x_3\text{-}x_2 & x_1\text{-}x_3 & x_2\text{-}x_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

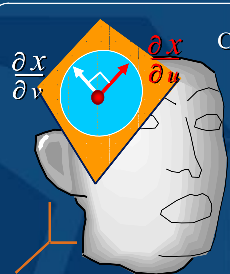
Parameterization
G coefficients - Previous Work

- **[Maillot 93]**
  - $\| G - I \|^2$
- **[Hormann 00] MIPS :**
  - $\| J \|_F \| J^{-1} \|_F = a/b + b/a = \text{trace}(G) / \det(J)$
- **[Sander 01] Stretch minimization :**
  - $L^2 = \sqrt{(a^2+b^2) / 2}$; $L^\infty = \max(a,b)$
- **Conformal Maps, Dirichlet Energy**
  - [Pinkall93],[Eck95],[Haker00],[Desbrun01], [Sheffer01]

Parameterization
other methods

- **Barycentric Maps [Floater 95], [Levy 98]**
- **Spectral methods, MDS [Zigelman]**
- **Gradient Regularization [Levy 01]**
- **Compatible triangulations [Gotsmann]**

Parameterization
Notion of conformality



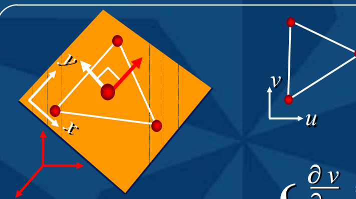
Conformal = locally isotropic

a = b

Laplace – Beltrami Δ

$$\frac{\partial x}{\partial v} \wedge N = \frac{\partial x}{\partial u}$$

Parameterization
Notion of conformality



Cauchy-Riemann: $\begin{cases} \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \end{cases}$

Parameterization
Link with Dirichlet Energy

$$E_C(u) + A_u(T) = E_D(u)$$

where:

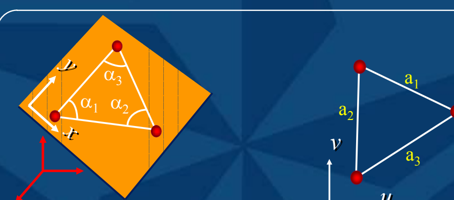
$E_D(u) = \frac{1}{2} . \int |\nabla u|^2$ *Dirichlet Energy*

$A_u(T) = \int \det(J_u)$ *Area of T*

$E_C(u) = \frac{1}{2} . \int || D^{90}(\partial u) - \partial v ||^2$ *Conformal Energy*

[Douglas31] [Rado30] [Courant50] [Brakke90]

Parameterization
The cotangent coefficients [Pinkall93]



$$E_D(u) = \frac{1}{2} . \int |\nabla u|^2 = \sum \cot(\alpha_i) . a_i^2$$

[Pinkall93],[Eck95],[Haker00],[Desbrun01]

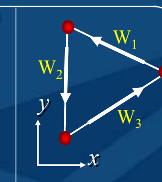
LSCM
Strategy

- **Sum of Squares ⇨ Gramm matrices**
- **Topo. Disc ⇨ Euler operators**
- **Similarity Invariance ⇨ $\mathbb{C}$**

LSCM
Cauchy-Riemann in a Δ : back to the roots (of -1)

$$\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y} = i\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)$$

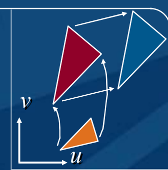
$$E_C(T) = \frac{1}{2A} \left| [\, W_1 \; W_2 \; W_3 \,] \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \right|^2$$



$W_k = (x_{k+2}-x_{k+1}) + i(y_{k+2}-y_{k+1})$; $U_k = u_k+iv_k$

LSCM
Cauchy-Riemann in a Δ : back to the roots (of -1)

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \xrightarrow{f} [\,W_1 \;\; W_2 \;\; W_3\,] \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$



$$\mathrm{Ker}(f) = \mathrm{Span}\left(\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \qquad R.e^{i\theta}.\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

LSCM
Matrix Form of the Criterion

$$E_C(\,[U_1,\ldots,U_n]^t\,) = \sum_T E_C(T)$$

$$= \overline{U}^t.\mathcal{C}.U$$

where $\mathcal{C} = \overline{M}^t M$

and $m_{i,j} = \begin{cases} W_{j,T_i} / \sqrt{2.A(T_i)} & \text{if } j \text{ in } T_i \\ 0 & \text{otherwise} \end{cases}$

LSCM
Removing degrees of freedom

$$C = \left\| \left[\begin{array}{c|c} M_f & M_p \end{array} \right] \begin{bmatrix} uf_1 \\ \vdots \\ uf_{nf} \\ \hdashline up_1 \\ \vdots \\ up_{np} \end{bmatrix} \right\|^2_H$$

$U = [U_f\,;\,U_p]^t$ $\qquad$ $M = [M_f\,;\,M_p]$

$E_C(U_f) = \| M_f.U_f + M_p.U_p \|^2_H$

LSCM
Back to ℝeality

$Ec(\mathbf{x}) = \| \mathbf{A}.\mathbf{x} - \mathbf{b} \|^2$ $\qquad$ $a+ib \equiv \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} Re(\mathbf{M}_f) & -Im(\mathbf{M}_f) \\ Im(\mathbf{M}_f) & Re(\mathbf{M}_f) \end{bmatrix}$$

$$\mathbf{b} = - \begin{bmatrix} Re(\mathbf{M}_p) & -Im(\mathbf{M}_p) \\ Im(\mathbf{M}_p) & Re(\mathbf{M}_p) \end{bmatrix} \begin{bmatrix} Re(U_p) \\ Im(U_p) \end{bmatrix}$$

LSCM
Critical point of E_c

$$Ec(\mathbf{x}) = \| \mathbf{A}.\mathbf{x} - \mathbf{b} \|^2$$

if $\mathbf{A}^t.\mathbf{A}$ is non-singular, the critical point x* is given by:

$\mathbf{x}^* = (\mathbf{A}^t.\mathbf{A})^{-1}.\mathbf{A}^t.\mathbf{b}$

Rem: $A^t.A$ is the Gramm matrix of the columns of A

{A is of maximum rank} => {$A^t.A$ is non-singular}

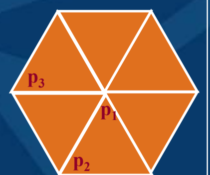
LSCM
Euler Operators

- -Two operators, *glue* and *join*
- -Preserve the maximal rank property of **A**

- -Vertices -> columns of A
- -Triangles -> rows of A

LSCM
The *join* operator

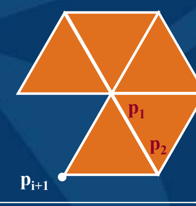
$$\left(\begin{array}{c} M_f^{(i)} \\ \hline W_{1,T}\ \ W_{2,T}\ \ W_{3,T}\ \ 0..0 \end{array}\right)$$



LSCM
The *glue* operator

$$\left(\begin{array}{c|c} M_f^{(i)} & 0 \\ \hline W_{1,T}\ \ W_{2,T}\ \ 0..0 & W_{i+1,T} \end{array}\right)$$

$\lambda_1 \quad \lambda_2 \quad .... \quad \lambda_{i+1}$



LSCM – other properties
Triangle Flips - erratum

- For internal vertices,
  LSCM = cotangent formula
- [Desbrun02] = LSCM
  ⇒Triangle flips may occur.
  Counter-example (see web-page)
  In practice: never observed

How to fix this:
- Constrained optimization as in [Sheffer]
- Rivara's subdivision as in [Desbrun]

LSCM
Other properties

- Natural border extrapolation
- Taken into account by quadratic form
- Independance to a similarity applied to the pinned vertices
- Independance to resolution

LSCM
Example: a huge chart (72500 trgls)