SIGGRAPH 2002

Linear Combination of Transformations

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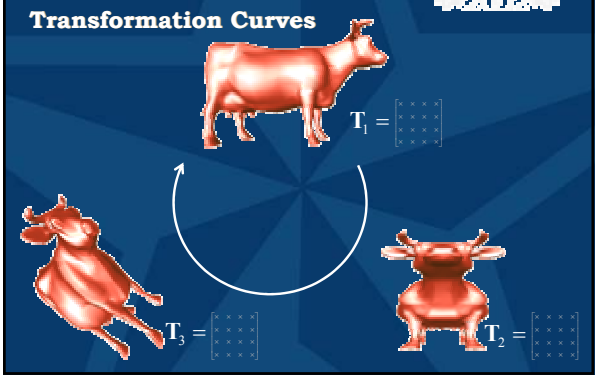
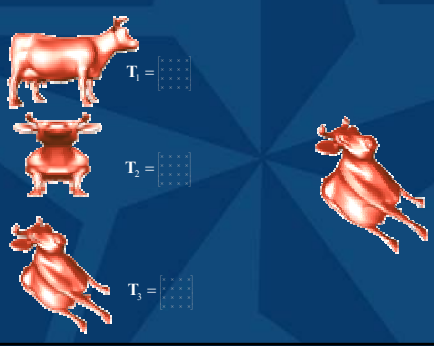
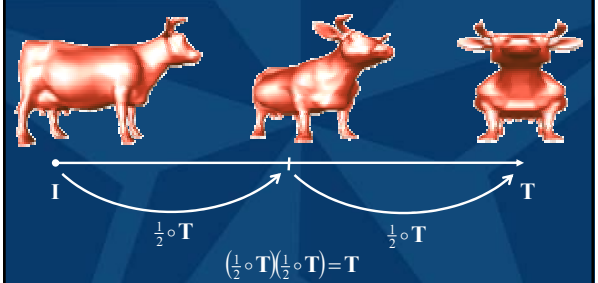
Motivation
Transformation Curves



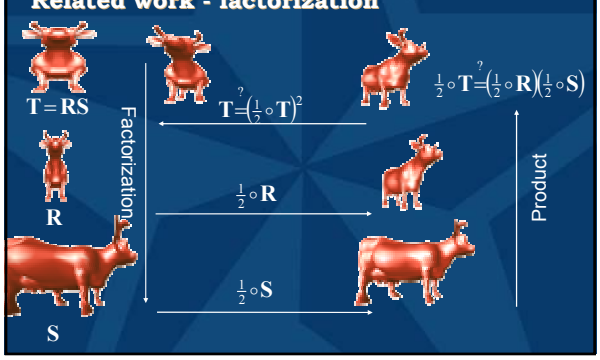
Motivation
Transformation Curves



Motivation
Basic problem & requirement



Motivation
Related work - factorization



Motivation
Requirements

- We need a scalar multiplication $\circ$ and a commutative addition $\oplus$
- Geometric requirement for scalar multiples:

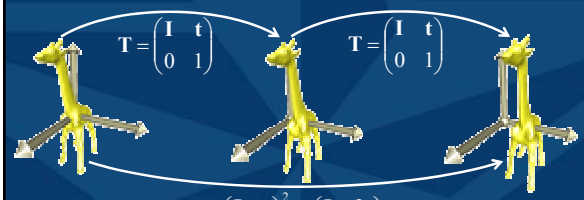
- Intuition for commutative addition:

$$A \oplus B = B \oplus A$$
$$A \oplus B = AB \Leftrightarrow AB = BA$$

Scalar multiples

- We expect $\left(\frac{1}{2}\circ\mathbf{T}\right)\left(\frac{1}{2}\circ\mathbf{T}\right)=\mathbf{T}$
- Based on matrix algebra this means: $\left(\frac{1}{2}\circ\mathbf{T}\right)=\mathbf{T}^{\frac{1}{2}}$ or generally $(s\circ\mathbf{T})=\mathbf{T}^{s}$
- Let's examine canonical transforms
  - Translation
  - Scaling
  - Rotation

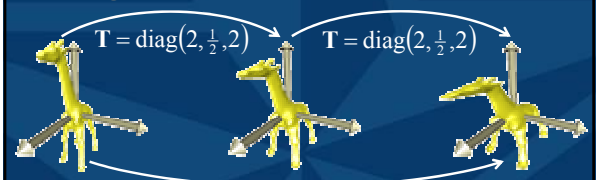
Scalar multiples
Translation



$$\mathbf{T}^2=\begin{pmatrix}\mathbf{I} & \mathbf{t}\\ 0 & 1\end{pmatrix}^2=\begin{pmatrix}\mathbf{I} & 2\mathbf{t}\\ 0 & 1\end{pmatrix}$$

Generally: $\begin{pmatrix}\mathbf{I} & \mathbf{t}\\ 0 & 1\end{pmatrix}^s=\begin{pmatrix}\mathbf{I} & s\mathbf{t}\\ 0 & 1\end{pmatrix}=s\circ\mathbf{T}$

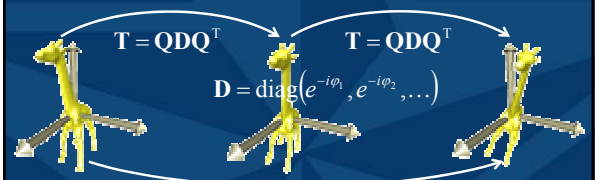
Scalar multiples
Scaling



$$\mathbf{T}^2=\left(\mathrm{diag}\left(2,\tfrac{1}{2},2\right)\right)^2=\mathrm{diag}\left(4,\tfrac{1}{4},4\right)$$

Generally: $s\circ\mathbf{T}=\mathrm{diag}\left(d_1^s,d_2^s,\ldots\right)$

Scalar multiples
Rotation



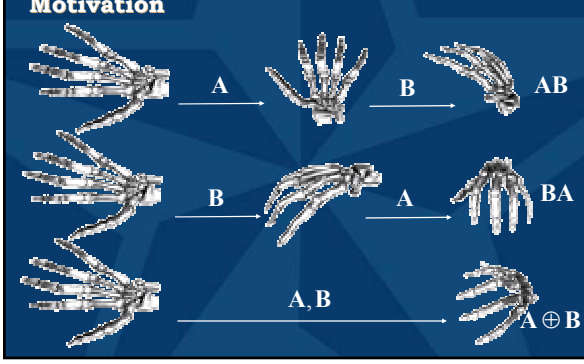
$$\mathbf{T}^2=\mathbf{QDQ}^{\mathrm{T}}\mathbf{QDQ}^{\mathrm{T}}=\mathbf{QD}^2\mathbf{Q}^{\mathrm{T}}$$

Generally: $s\circ\mathbf{T}=\mathbf{QD}^s\mathbf{Q}^{\mathrm{T}},\ \mathbf{D}^s=\mathrm{diag}\left(e^{-i(s\varphi_1)},e^{-i(s\varphi_2)},\ldots\right)$

Scalar multiples
General case

- Define $s\circ\mathbf{T}=\mathbf{T}^s$
- Has every real square matrix $\mathbf{T}$ unique real rational powers?
- Yes, if $\mathbf{T}$ has no negative real eigenvalues
- Some consequences/properties:
  - $0\circ\mathbf{T}=\mathbf{T}^0=\mathbf{I}\qquad 1\circ\mathbf{T}=\mathbf{T}^1=\mathbf{T}$
  - $(r+s)\circ\mathbf{T}=\mathbf{T}^{(r+s)}=\mathbf{T}^r\mathbf{T}^s=\mathbf{T}^s\mathbf{T}^r$

Commutative addition
Motivation

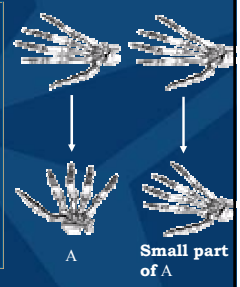


Commutative addition
Implementation

- Take "small parts" of A and B and apply them alternately
- "Small parts" = small scalar multiple

$$\tfrac{1}{n} \circ \mathbf{A} = \mathbf{A}^{1/n}, \tfrac{1}{n} \circ \mathbf{B} = \mathbf{B}^{1/n}$$

- Result $\left(\mathbf{A}^{1/n}\mathbf{B}^{1/n}\right)^n$



Commutative addition
Rationale & Definition

- For $n \to \infty$ we expect $\mathbf{A}^{1/n}, \mathbf{B}^{1/n} \to \mathbf{I}$
- $\left(\mathbf{A}^{1/n}\mathbf{B}^{1/n}\right)^n$ and $\left(\mathbf{B}^{1/n}\mathbf{A}^{1/n}\right)^n$ differ by $\mathbf{A}^{1/n}, \mathbf{B}^{1/n}$
- Definition: $\mathbf{A} \oplus \mathbf{B} = \lim_{n\to\infty}\left(\mathbf{A}^{1/n}\mathbf{B}^{1/n}\right)^n$

$\mathbf{A}$ $\mathbf{A}^{1/2}$ $\mathbf{A}^{1/4}$ $\mathbf{A}^{1/8}$ $\mathbf{A}^{1/16}$ $\to$ $\mathbf{I}$

Commutative addition
Questions

$$\mathbf{A} \oplus \mathbf{B} = \lim_{n\to\infty}\left(\mathbf{A}^{1/n}\mathbf{B}^{1/n}\right)^n$$

- Does the limit exist?
- Is it real, if A and B are real?
- Is the operation commutative?
- Yes to all, under reasonable conditions!
- Property: $\mathbf{A} \oplus \mathbf{B} = \mathbf{AB}$ iff $\mathbf{AB} = \mathbf{BA}$

Computation
Matrix Algebra Tools

- Matrix exponential: $e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$
- Matrix logarithm: $e^{\mathbf{A}} = \mathbf{B} \Leftrightarrow \mathbf{A} = \log \mathbf{B}$

$\longrightarrow$ $s \circ \mathbf{A} = e^{s \log \mathbf{A}}$

- Lie Product: $e^{\mathbf{A}+\mathbf{B}} = \lim_{n\to\infty}\left(e^{\mathbf{A}/n}e^{\mathbf{B}/n}\right)^n$

$\longrightarrow$ $\mathbf{A} \oplus \mathbf{B} = e^{\log \mathbf{A} + \log \mathbf{B}}$

Computation
Implementation

- Linear combination:

$$\bigoplus_i s_i \circ \mathbf{T}_i = e^{\sum_i s_i \log \mathbf{T}_i}$$

$\left\{\mathbf{T}_i = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}\right\} \xrightarrow{\text{Transparent OO Impl.}} \exp \mathbf{R} = \bigoplus_i s_i \circ \mathbf{T}_i$

Init (30µs)

$\{\log \mathbf{T}_i\} \xrightarrow{\text{Standard linear combination}} \mathbf{R} = \sum_i s_i \log \mathbf{T}_i$

Display (3µs)

Applications

- Smooth animation curves
- Factoring transforms to create application specific transform spaces
- Analysis of transform animations

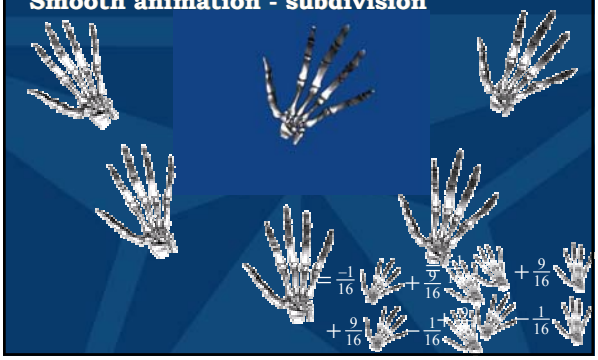
Applications
Animations – linear interpolation

A =

B =

$$\mathbf{C}(t) = (1-t)\circ\mathbf{A} \oplus t\circ\mathbf{B} = \exp((1-t)\log\mathbf{A} + t\log\mathbf{B})$$

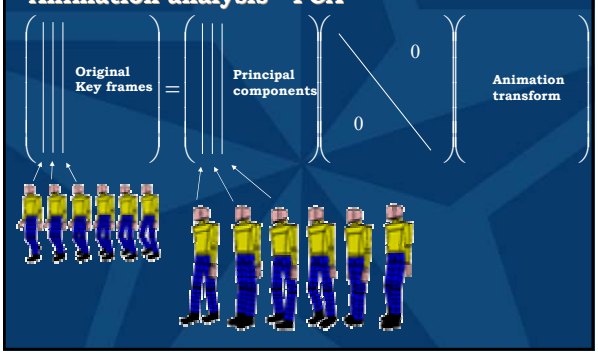
Applications
Smooth animation - subdivision



Applications
Animation analysis

- **Animation**
  - 17 joints
  - 17 4x4 matrices
  - changing over time
  - asynchronously

Applications
Animation analysis - PCA



Applications
Animation using 1 PC

$s(t)\circ$

$=$

17 4x4 (log) matrices

Applications
Animation using 2 PCs

$s_1(t)\circ$

$\oplus s_2(t)\circ$

$=$

Applications
Animation using 5 PCs

$s(t) \bullet$ $=$

Conclusions

- **Definition of**
  - Scalar multiples and
  - Commutative addition
  - Based on the transformation matrices
- **Easy to implement and use**
  - Simple re-use of existing techniques for linear spaces
- **Quickly computed**
- **Works in any dimension**

Thank you!