Reliable Two-Dimensional Graphing Methods for Mathematical Formulae with Two Free Variables

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An Equation of a Circle

$$x^2+y^2=36$$

Example Inequality

$$888\sin(2x+y)\sin(2x-y)\sin(x+2y)\sin(x-2y)>y-x$$

Example Equation

$$\sin\sin\sqrt{|x^2\sin x+y^2\cos x|}+\sin x=\cos\sin\sqrt{|x^2\cos y+y^2\sin y|}+\cos y$$

Example Inequality

$$1.9\cos\left(69q+2\left\lfloor\frac{27r}{p}\right\rfloor\right)<\frac{2}{p}\operatorname{mod}(r,p)+\sin\left(q+p\left\lfloor\frac{r}{p}\right\rfloor\right)-1$$

$$r<3p$$

Example Equation

$$k=\left(\left\lfloor\frac{|x|}{4}\right\rfloor+3\right)m$$

$$j=\left\lfloor\frac{k}{2p}\operatorname{angle}(\operatorname{mod}(x,4)-2,\operatorname{mod}(y,4)-2)\right\rfloor$$

$$\left(\operatorname{mod}(x,4)-2-d\cos\frac{2pj}{k}\right)\left(D\sin\frac{2p(j+1)}{k}-d\sin\frac{2pj}{k}\right)=\left(D\cos\frac{2p(j+1)}{k}-d\cos\frac{2pj}{k}\right)\left(\operatorname{mod}(y,4)-2-d\sin\frac{2pj}{k}\right)$$

$$|\operatorname{mod}(x,4)-2|+|\operatorname{mod}(y,4)-2|>\frac{3}{5}$$

$$d=\frac{5}{3}\operatorname{signum}(\gcd(j+1,m)-1)$$

$$D=\frac{5}{3}\operatorname{signum}(\gcd(j+2,m)-1)$$

$$m=1+\left\lfloor\frac{|y|}{4}\right\rfloor$$

Example Inequality

$$|\cos(4\cos\min(\sin x+y, x+\sin y))-\cos(4\sin\max(\sin y+x, y+\sin x))|$$
$$<|\cos(\cos\min(\sin x+y, x+\sin y))-\cos(\sin\max(\sin y+x, y+\sin x))|-0.02$$

Example Equation

$$\sin\left(2^{\lfloor y\rfloor}x\pm\frac{p}{4}(y-\lfloor y\rfloor)-\frac{p}{2}\right)=0$$

Example Inequality

$$\sin\sin[x[\sin y-\cos x]]<\cos\cos[y[\cos x-\sin y]]$$

Example Inequality

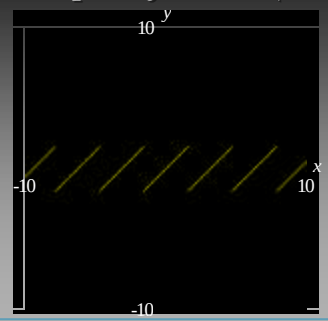
$$\sin(\min([x+y]\sin[y-x],[y-x]\sin[x+y]))$$
$$<\cos(\max([x+y]\cos[y-x],[y-x]\cos[x+y]))$$
$$+\frac{1}{6400000}\left[8x^2+4(y-3)^2\right]^3+\frac{1}{15}[6-y]$$

Isn't this already solved?

There are many utilities for doing this:

- **Computer Algebra Systems**
  - Mathematica, Maple, …
- **Graphing Calculators**
  - Hewlett-Packard, Texas Instruments, …
- **Graphing Software**
  - Curvus Pro, IAsolve, GraphingCalculator, …

Correct Graph of *y*=Arctan(tan(*x*))



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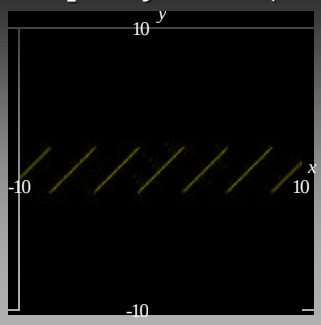

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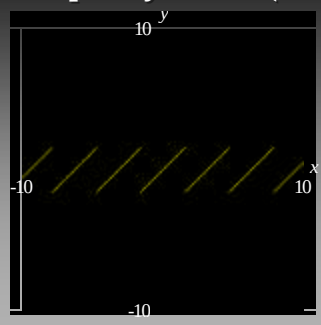

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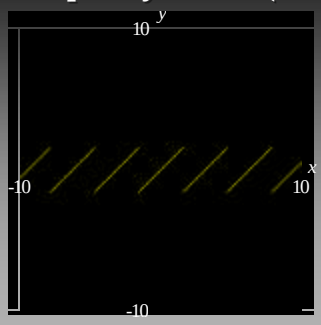

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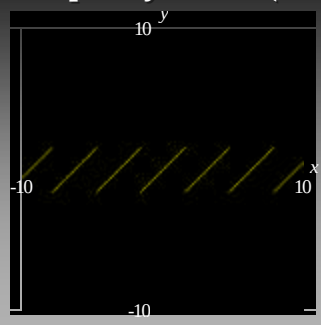

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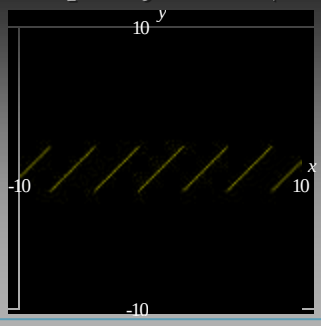
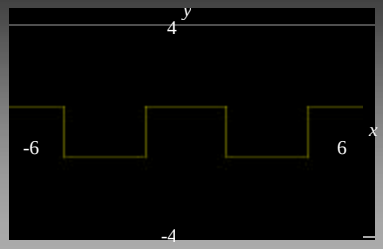


Correct Graph of $y=\sqrt[9]{\cos x}$

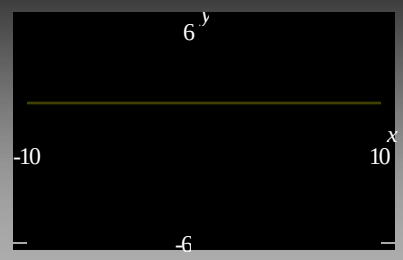


Floating-Point Arithmetic

- Many graphing programs use floating-point arithmetic to evaluate formulae
- Floating-point arithmetic is not exact

1.000 ¸ 3.000 ® 0.3333
0.3333 ´ 3.000 ® 0.9999

Correct Graph of $y=(2^{55}+2+\sin x-2^{55})-\sin x$



Connect-the-Dots Graphing

Problems:

- Not all dots should be connected
- Dots may be far from the curve

Connect-the-Dots Graphing

Fundamental Problem:

- We haven't defined the graph's semantics

Graph Semantics

A pixel is:

Yellow Û Solution Exists
Black Û No Solution Exists

Example Equation

$$\sin((x \pm \sin y)(\sin x \pm y))$$
$$= \cos\sin((\sin x \pm \cos y)(\sin y \pm \cos x))$$

Example Inequality

$$\sin 30\sqrt{\left(\sin\frac{x-4}{3}\right)^2+\left(\cos\frac{y+6}{3}\right)^2}+$$

$$\sin 20\sqrt{\left(\sin\frac{x-3}{2}\right)^2+\left(\cos\frac{y+5}{2}\right)^2}>0$$

Example Inequality

$$\lfloor \gcd(x,y)\rfloor+(\gcd(\lfloor x\rfloor,\lfloor y\rfloor)-2)\geq 0$$

Unfortunate Reality

- **This naïve goal is impossible since graphing, as formalized, is not computable**

Practical Graph Semantics

A pixel is:

Yellow Þ Solution Exists
Black Þ No Solution Exists
Red Þ Maybe, Maybe Not

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Reliable Graphing

- We now have a well-defined problem
- But how do we evaluate formulae?

Formula Evaluation

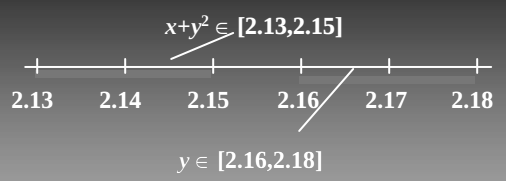
- Use interval arithmetic to evaluate formulae
- Interval arithmetic provides guaranteed bounds on accuracy

Interval Arithmetic

- Compute using lower and upper bounds

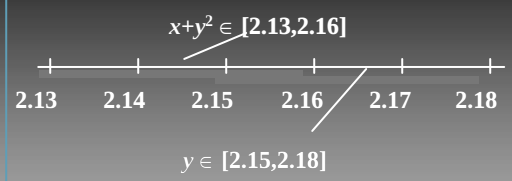
á1.00,1.00ñ ¸ á3.00,3.00ñ ® á0.333,0.334ñ
á0.333,0.334ñ ´ á3.00,3.00ñ ® á0.999,1.01ñ

Interval Comparisons



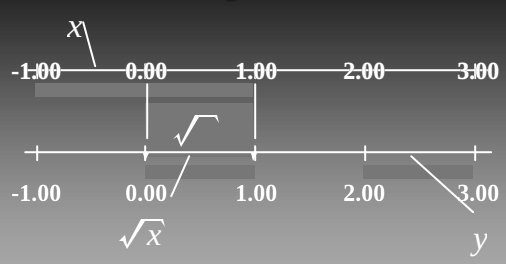
Is $x+y^2<y$? Yes.

Interval Comparisons



Is $x+y^2<y$? Maybe.

Domain Tracking



Is $\sqrt{x}<y$? Yes.

Domain Tracking

Is $\sqrt{x}<y$? Maybe.

Iterative Graphing Algorithm

- Begin with a solid red display
- Keep a list of uncertain regions
- Evaluate uncertain regions, coloring and subdividing as appropriate

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Pixel Boundaries

- True pixel boundaries may not be floating-point numbers

Example Inequality

$$(\cos y \sin x - \cos x)(\cos x \sin y - \cos y) > 0$$

Example Inequality

$$\cos 18x + \cos 9(x-\sqrt{3}\,y) + \cos 9(x+\sqrt{3}\,y) + 3 < 0 \quad \begin{cases} \frac{3}{4} - \frac{1}{15}\sqrt{(x+4)^2+(y-3)^2} & \text{if } (x+1)^2+(y-1)^2 < 25 \\ 0.65 + \frac{1}{p}\,\text{Arctan}\left[\sqrt{\frac{(x-1)^2}{30} + \frac{(y+1)^2}{9}} - 1\right] & \text{if } (x+1)^2+(y-1)^2 > 25 \end{cases}$$

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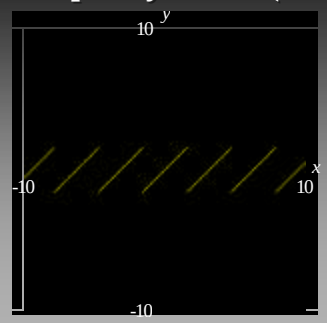

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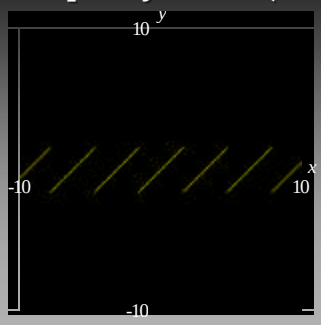
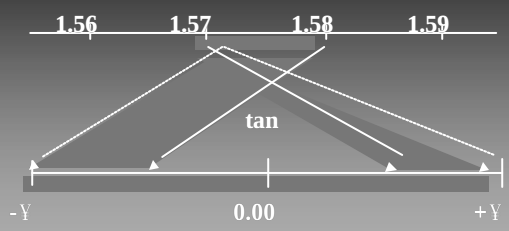
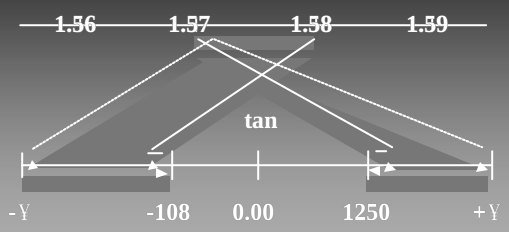


Interval Arithmetic



Interval Sets



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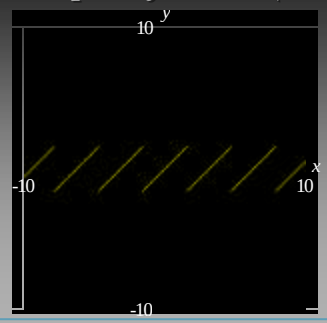
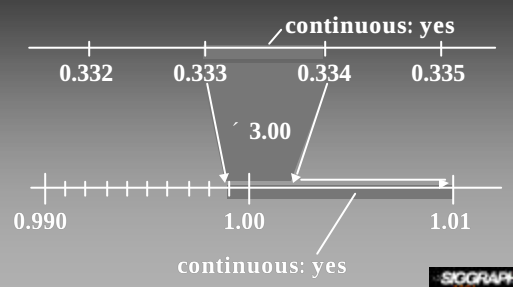


Continuity Tracking



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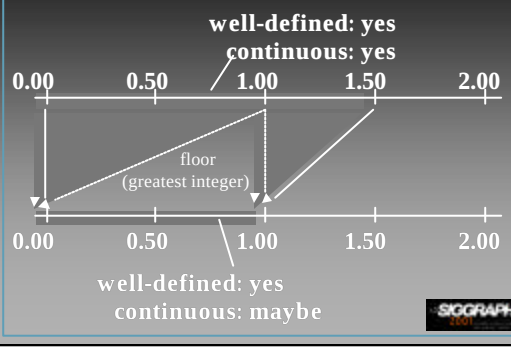
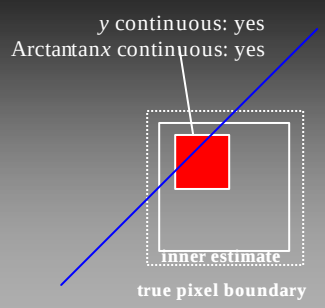


Finding Solutions on Curves



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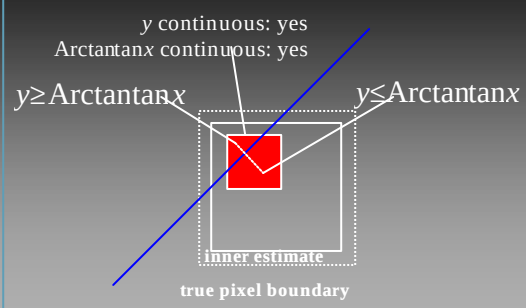
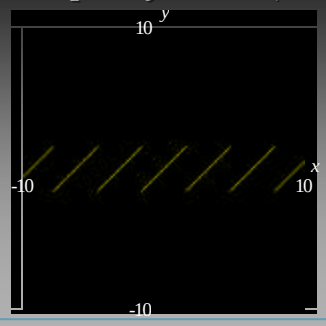


Correct Graph of y=Arctan(tan(x))



Example Equation

$$|x\cos x + y\sin y| = x\cos y - y\sin x$$

Example Inequality

$$\mathrm{mod}\left(\left\{\begin{array}{lll} 892 & \text{if} & 30\max\left(\left|Y-\frac{1}{2}\right|,\left|X-\frac{1}{2}\right|+\left|Y-\frac{1}{2}\right|-\frac{1}{4}\right)<1 \\ 365 & \text{if} & 30\max\left(\left|Y-\frac{1}{10}\right|,\left|X-\frac{1}{2}\right|+\left|Y-\frac{1}{10}\right|-\frac{1}{4}\right)<1 \\ 941 & \text{if} & 30\max\left(\left|Y-\frac{9}{10}\right|,\left|X-\frac{1}{2}\right|+\left|Y-\frac{9}{10}\right|-\frac{1}{4}\right)<1 \\ 927 & \text{if} & 30\max\left(\left|X-\frac{4}{5}\right|,\left|Y-\frac{7}{10}\right|+\left|X-\frac{4}{5}\right|-\frac{1}{8}\right)<1 \\ 881 & \text{if} & 30\max\left(\left|X-\frac{1}{5}\right|,\left|Y-\frac{7}{10}\right|+\left|X-\frac{1}{5}\right|-\frac{1}{8}\right)<1 \\ 325 & \text{if} & 30\max\left(\left|X-\frac{1}{5}\right|,\left|Y-\frac{3}{10}\right|+\left|X-\frac{1}{5}\right|-\frac{1}{8}\right)<1 \\ 1019 & \text{if} & 30\max\left(\left|X-\frac{4}{5}\right|,\left|Y-\frac{3}{10}\right|+\left|X-\frac{4}{5}\right|-\frac{1}{8}\right)<1 \end{array}\right\}(2)^{-\left\lfloor \mathrm{mod}\left(10^{\left\lfloor y \right\rfloor}\cdot\left\lfloor 1.25x \right\rfloor,10\right)\right\rfloor},2\right)\geq 1$$

$$X = \mathrm{mod}(x, 0.8) + 0.1$$
$$Y = \mathrm{mod}(y, 1)$$
$$0.8\lfloor -\log y \rfloor < x < 0$$

Example Inequality

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Conclusion

- **Most graphing programs are not reliable**
  - Reliable graphing programs do exist (GrafEq)
- **Red pixels are useful**
- **Be careful when using interval arithmetic**
  - Keeping track of the mathematical properties of evaluated formulae is possible and useful

Future Work

- **Use other colors besides red**
  - Display topological information
- **Tackle a larger class of formulae**
  - integration, differentiation, iteration, …
- **Animation**
  - visualize role of parameters
- **3D**

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- **Alain Fournier;**
- **my supervisor, Eugene Fiume;**
- **John Hughes and the other paper reviewers, for their helpful comments.**

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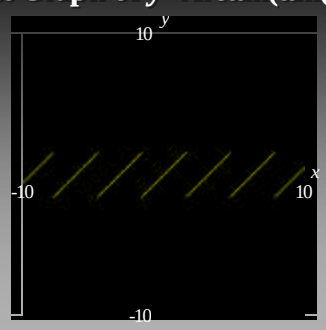
GrafEq:

- www.peda.com/grafeq
- Creative Applications Lab 1PM-2PM Today

Example Inequality

$$\sin(5(\sin x - \sin y)) > \sqrt{|\sin x + \sin y|}$$

Example Equation

$$\mathrm{mod}(\sin x, \cos y) = \mathrm{mod}(\sin y, \cos x)$$

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