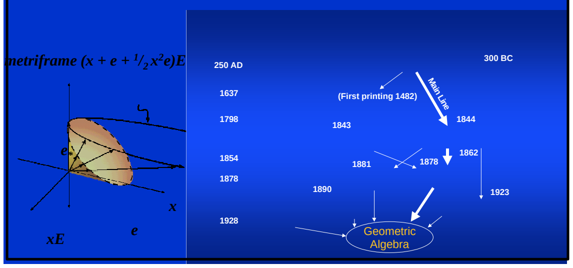
Introduction to Geometric Algebra

Course # 53

Organizers: Ambjørn Naeve
Alyn Rockwood



Course Speakers

Alyn Rockwood (Colorado School of Mines)
David Hestenes (Arizona State University)
Leo Dorst (University of Amsterdam)
Stephen Mann (University of Waterloo)
Joan Lasenby (Cambridge University)
Chris Doran (Cambridge University)
Ambjørn Naeve (Royal Technical Institute of Stockholm)

[a] mathematician is a Platonist on weekdays and a Formalist on Sundays. That is, when doing mathematics he is convinced that he is dealing with objective reality … when challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it after all. ~P. Davis

Mathematics is Language

	Primitive	
	Nouns	**Verbs**
Vector Algebra	scalar, vector	scalar, dot & cross products, scalar & vector addition gradient, curl, …
Complex Analysis	real, imaginary	addition, multiplication, conjugation, …
Synthetic Geometry	points, line, circles …	intersection, union, …

A Redundant Language

- Synthetic Geometry
  Coordinate Geometry
  Complex Numbers
  Quaternions
  Vector Analysis
  Tensor Analysis
  Matrix Algebra
  Grassmann Algebra
  Clifford Algebra
  Spinor Algebra
  …

Geometric Concepts

- Consequences
  - Redundant learning
  - Complicates knowledge access
  - Frequent translation
  - Lower concept density, i.e., theorems / definitions

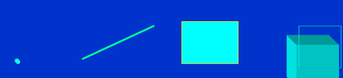
A language for geometry

Hermann Grassmann 1809 - 1877 (Our Hero)

Geometric Concepts → **Algebraic Language**

Properties of nouns

- Grade - dimension



A language for geometry

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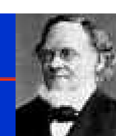

Geometric Concepts → **Algebraic Language**

Properties of nouns

- Grade - dimension
- Direction - orientation, attitude, how it sits in space

A language for geometry

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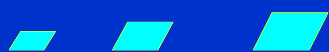
Geometric Concepts → Algebraic Language

Properties of nouns

- Grade - dimension
- Direction - orientation, attitude, how it sits in space
- Magnitude - scalar



A language for geometry

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Geometric Concepts → Algebraic Language

Properties of nouns

- Grade - dimension
- Direction - orientation, attitude, how it sits in space
- Magnitude - scalar
- Sense - positive/negative, up/down, inside/outside

Geometric Algebra

D. Hestenes, *New Foundations for Classical Mechanics*, Kluwer Academic Publishers, 1990

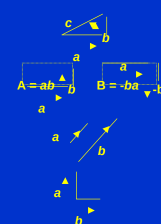
Geometric Concepts → Algebraic Language

Primitive nouns

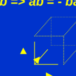
• Point	α	scalar	grade 0
• Vector	***a***	directed line	grade 1
• Bi-vector	**A**	directed plane	grade 2
• Tri-vector	**T**	directed volume	grade 3
• Etc.			

Verbs

Geometric Concepts → Algebraic Language



- Addition — *c* = *a* + *b* = *b* + *a*
- Multiplication — A = -B
- Commutivity — *a* || *b* => *ab* = *ba*
- Anticommutivity — *a* ^ *b* => *ab* = - *ba*
- Associativity — *a* ^ *b* ^ *c* => (*ab*) *c* = *a* (*bc*) = T



- and others

Prepositions

William Kingdon Clifford 1845 - 1879 (Another Hero)

- **Complex analysis**

  Addition defines relation, i.e. *a* + *i b* ° (*a*, *b*)

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- **Clifford’s “geometric product” for vectors**

  ab = ***a*** ˑ ***b*** + ***a*** Ù ***b***

Prepositions

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  $ab = a \cdot b + a \wedge b$

  scalar (dot product) **bi-vector (exterior product)**

Prepositions

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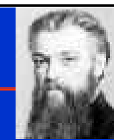

- **Complex analysis**

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- **Clifford's "geometric product" for vectors**

  $ab = a \cdot b + a \wedge b$

  scalar (dot product) **bi-vector (exterior product)**

  prepositional add

Geometric Algebra

Nouns <u>*k*-vectors</u> (scalar, vector, bi-vector …)

• Point	α	scalar	grade 0
• Vector	$\mathbf{a}$	directed line	grade 1
• Bivector	$\mathbf{A} = \mathbf{a} \wedge \mathbf{b}$	directed plane	grade 2
• Trivector	$\mathbf{T} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$	directed volume	grade 3
• *k*-vector	$V = \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_k$		grade *k*

Geometric Algebra

Nouns <u>*k*-vectors</u> (scalar, vector, bi-vector …)

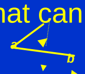
and <u>multivectors</u> (sums of *k*-vectors)

• Point	α	scalar	grade 0
• Vector	$\mathbf{a}$	directed line	grade 1
• Bivector	$\mathbf{A}$	directed plane	grade 2
• Trivector	$\mathbf{T}$	directed volume	grade 3
• …			
• Multivector	$\mathbf{M}$	sum of *k*-vectors	mixed grade
	$(M = \alpha + a + A + T + \ldots)$		

The Geometric Product

Verbs - what can two vectors do?

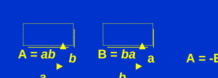
- Project



The Geometric Product

Verbs - what can two vectors do?

- Project
- Define bi-vector



The Geometric Product

Verbs - what can two vectors do?

- Project
- Define bi-vector
- Commute   a || b => ab = ba

The Geometric Product

Verbs - what can two vectors do?

- Project
- Define bi-vector
- Commute
- Anti-commute   a ^ b => ab = - ba

The Geometric Product

What can two vectors do?

- Project
- Define bi-vector   A = ab   B = -ba   A = -B
- Commute   a || b => ab = ba
- Anti-commute   a ^ b => ab = - ba

ab = a· b + a Ù b

The Geometric Product

… is more basic!!

Define vector dot product in terms of GP

a · b = 1/2 (ab + ba) ← **scalar**

Define vector wedge product in terms of GP

a Ù b = 1/2 (ab - ba) ← **bivector**

⇒ **(a · b + a Ù b = ab)**

Examples

Reflection

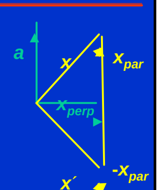
For $a^2 = 1$, $-a\,x\,a = -a\,(x_{par} + x_{perp})\,a$

$= -\,(a\,x_{par} + a\,x_{perp})\,a$

GA → $= -\,(x_{par}\,a - x_{perp}\,a)\,a$

$= -\,(x_{par} - x_{perp})\,a^2$

$= -\,x_{par} + x_{perp} = x'$



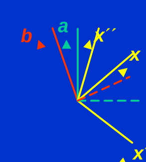
Examples

Rotations (b^2=1)

$x'' = -b\,x'\,b = -b\,(-a\,x\,a)\,b = (b\,a)\,x\,(a\,b)$

$(b\,a)\,x\,(a\,b)$ **rotates** x

through 2 Ð ab



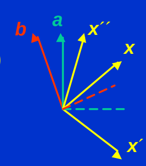
Examples

Rotations (any vectors a, b)

Let $R = ab$, define <u>reverse</u> $R^{\sim} = ba = (ab)^{\sim}$

then

$x'' = -b\,x'\,b = -b\,(-a\,x\,a)\,b = (b\,a)\,x\,(a\,b)$



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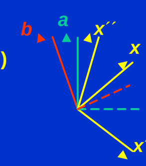


General form of rotation:

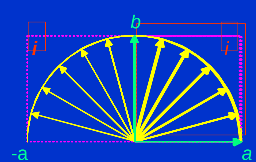
$\boxed{x'' = R^{\sim}\,x\,R}$

Examples

Let $a \bullet b = 0$ and $a^2 = b^2 = 1$, define $\boldsymbol{i} = a\,b = -b\,a$

$\boldsymbol{i}$ is an operator:

$a\,\boldsymbol{i} = a\,(a\,b) = a^2\,b = b$
rotates a by 90 degrees to b
$b\,\boldsymbol{i} = (a\,\boldsymbol{i})\,\boldsymbol{i} = a\,\boldsymbol{i}^2 = -a$
rotates a twice, giving $\boldsymbol{i}^2 = -1$



Examples

Let $a \cdot b = 0$ and $a^2 = b^2 = 1$, define $\boldsymbol{i} = a\,b = -b\,a$

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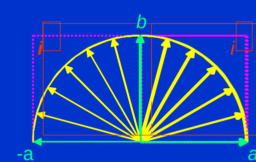

Bivectors rotate vectors (!)

Recapitulation

- Graded elements with sense, direction and magnitude
- Addition - verb and preposition form
- Geometric product is sum of lower and higher grades
- Dot and Wedge products defined by GP
- Two-sided vector multiplication reflects vectors
- Bivector multiplication rotates vectors
- Special unit bivector ***I*** **(pseudoscalar)**

Axioms

1. Algebra with non-commutative multiply

Think of matrix algebra

Axioms

1. Algebra with non-commutative multiply
2. Scalar multiplication commutes *l* A = A *l*
3. For vector ***a***² = |***a***|² ³ *0*, a scalar
4. a • A_k is a k-1 vector and a Ù A_k is a k+1 vector
   where a • A_k = ½ (aA_k – $(-1)^k$ A_k a
   and a Ù A_k = ½ (aA_k + $(-1)^k$ A_k a)

Differentiate elements of different grade

Axioms

1. Algebra with non-commutative multiply
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Generalizes dot and wedge products

Axioms

Truncates space to k dimensions

5. a Ù A_k = 0 for a k-dimensional space

Axioms

1. Non-commutative algebra – add and multiply
2. Scalar multiplication commutes *l* A = A *l*
3. For vector ***a***² = |***a***|² ³ *0*, a scalar
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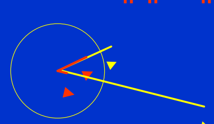
5. a Ù A_k = 0 for a k-dimensional space

Axiom Implications

1. Algebra with non-commutative multiply

Vector inverse x^{-1} = x / ||x|| for ||x|| ¹ 0



Axiom Implications

One equation yields separate graded equations:

V = α + *a* + B + T ⇒

$$\langle V\rangle_0 = \alpha$$
$$\langle V\rangle_1 = a$$
$$\langle V\rangle_2 = B$$
$$\langle V\rangle_3 = T$$

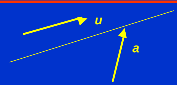
Where $\langle V\rangle_k$ is the *k*-vector part of V

…and many other algebraic devices

Example Algebra

Straight Lines

(x-a) Ùu = 0 defines line



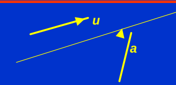
Example Algebra

Straight Lines

(x-a) Ùu = 0 defines line

x Ùu = x Ùa = M, a bivector

(x Ùu)u^{-1} = Mu^{-1} (division by vector!)



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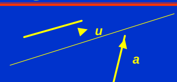
(x Ùu) • u^{-1} + (x Ùu) Ù u^{-1} = Mu^{-1} (expansion of GP)

(x Ùu) • u^{-1} + 0 = Mu^{-1} (wedging parallel vectors)



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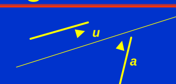
(x Ùu) • u^{-1} + 0 = Mu^{-1} (wedging parallel vectors)

x – (x • u) u^{-1} = Mu^{-1} (Laplace reduction theorem)

x = (M + x • u) u^{-1}

= (M + a) u^{-1}

Parametric form for fixed M and u.



Example Algebra

Straight Lines

(x-a) Ùu = 0 defines line

x = (M + a) u^{-1}

Parametric form for fixed M and u.

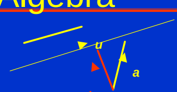
let d = Mu^{-1}

x = d + a u^{-1}

Since d • u = Mu^{-1} • u

d • u = 0 (grade equivalence)

d is orthogonal to u



Representations

Complex analysis
Duality
Quaternions
Vector algebra
Spherical geometry

Let $z = ab = a \cdot b + a$ Ù b ($a^2 = b^2 = 1$)

Let $z^\dagger = ba = (ab)^\dagger$ (reverse = conjugate)

Since $a \cdot b = ½(ab + ba) = ½(z + z^\dagger)$

$= Re\ z = \lambda\cos\theta$

and $a \wedge b = ½(ab - ba) = ½(z - z^\dagger)$

$= Im\ z = \lambda i\sin\theta$

then $z = \lambda(\cos\theta + i\sin\theta) = \lambda e^{i\theta}$

The shortest path to truth in the real domain often passes through the complex domain

Hadamard

Representations

Complex analysis
Duality
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Spherical geometry

Let s_1, s_2 and s_3 be orthonormal basis vectors then

$\{1,\ s_1, s_2, s_3,\ s_1s_2, s_1s_3, s_2s_3,\ s_1s_2s_3\}$

is a basis for the geometric algebra over R^3

Representations

Complex analysis
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Let s_1, s_2 and s_3 be orthonormal basis vectors then

$\{1,\ s_1, s_2, s_3,\ s_1s_2, s_1s_3, s_2s_3,\ s_1s_2s_3\}$

Scalar vector bivector trivector

Representations

Complex analysis
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Let s_1, s_2 and s_3 be orthonormal basis vectors then

Let $I = s_1s_2s_3$, the pseudoscalar

What are: $(s_1s_2s_3)^2 = I^2$? $I s_1$? $I s_1 s_2$?

Representations

Complex analysis
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$(s_1s_2s_3)^2 = I^2 = -1$

$I s_1$?

$I s_1 s_2$?

Representations

Complex analysis
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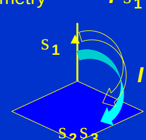
$(s_1s_2s_3)^2 = I^2 = -1$

$I s_1 = s_2 s_3$

transforms s_1 to s_2s_3

$I s_1 s_2 = - s_3$

transforms $s_1 s_2$ to $- s_3$



$I s_1 = s_2 s_3$

Representations

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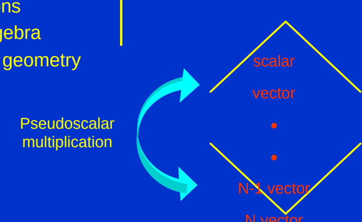
In general:



Representations

Complex analysis
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Let $i = s_2 s_3$

$j = - s_3 s_1$

$k = s_1 s_2$

then

$i^2 = j^2 = k^2 = -1$ **and** $i\,j\,k = -1$

Hamilton's equations for quaternions!

Representations

Complex analysis
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If (s, q_1, q_2, q_3) **is a quaternion**

then

$R = s + i\,q_1 + j\,q_2 + k\,q_3$

scalar **bivector**

Is a general rotor in GA

Recall $x' = RxR^{\sim}$

Note: i, j, k **are bivectors!**

Representations

Complex analysis
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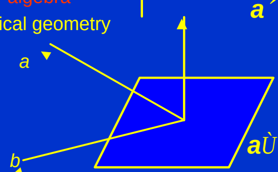
a´ b = i aÙ b



aÙ b = -i a´ b

Representations

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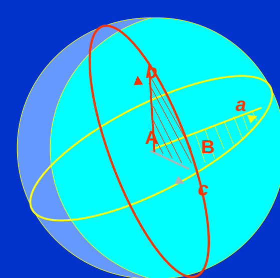

A = bc

B = ca

C = AB?

Representations

Complex analysis
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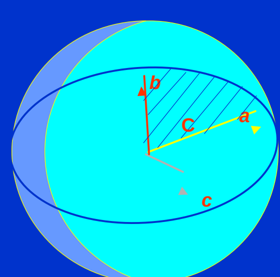


A = bc

B = ca

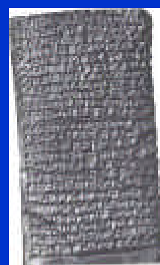
C = AB **=bcca = ba**

Advantages of GA

- **Unifying**
  - compact knowledge, enhanced learning, eliminates redundancies and translation
- **Geometrically intuitive**
- **Efficient**
  - reduces operations, coordinate free, separation of parts
- **Dimensionally fluid**
  - equations across dimensions

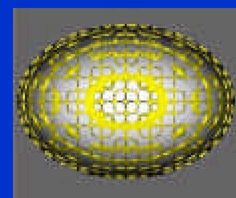
Advantages of GA

- **Unifying**
- **Geometrically intuitive**
- **Efficient**
- **Dimensionally fluid**
- **A better language**

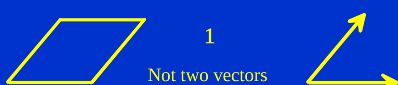


$(x+y)/2 \pm [(x+y)^2/4 - xy]^{1/2}$

Fini



Bivectors



Examples: