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Computational Fluid Dynamics

- over 50 years of existing technology for numerical simulation
- large community of scientists pushing the frontier to solve new and challenging problems
- they need accurate predictive results that aid in both understanding natural phenomena and controlling it

Technology Transfer

- draw upon both traditional and **new** methods in order to simulate natural phenomena for computer graphics (special effects for television and film)

Some Core Technology

level set methods - model the location of an interface even in the presence of extreme topological changes (merging and pinching) - low computational cost

ghost fluid method - models the physical boundary conditions at an interface - low computational cost

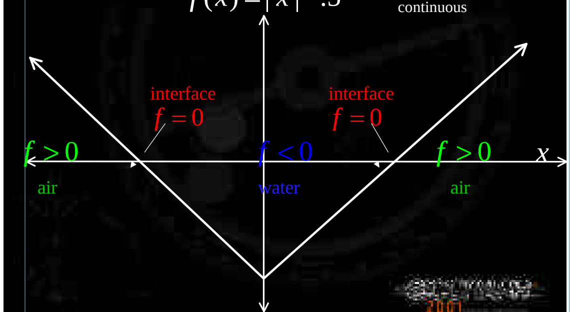
vorticity confinement - models small scale turbulence details in important "interfacial" regions of the flow - low computational cost

Why interfaces? that's what we see

Why low computational cost? we don't have 10^5 CPU's

Level Set Method (implicit surface)

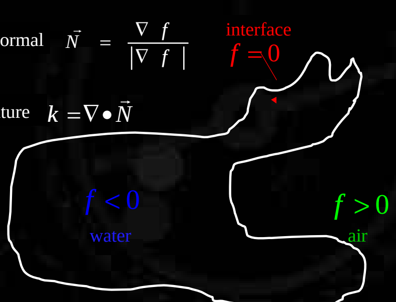
$f(x) = |x| - .5$ signed distance function continuous



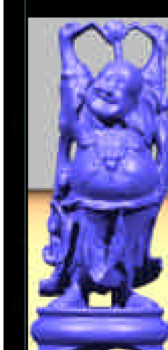
Level Set Method

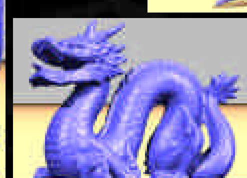
unit normal $\vec{N} = \frac{\nabla f}{|\nabla f|}$

curvature $k = \nabla \bullet \vec{N}$



Level Set Method





Level Set Method – "dynamic" implicit surface

move the interface $f_t + \vec{V} \bullet \nabla f = 0$

interface velocity

maintain signed distance $f_t + S(f_o)[|\nabla f| - 1] = 0$

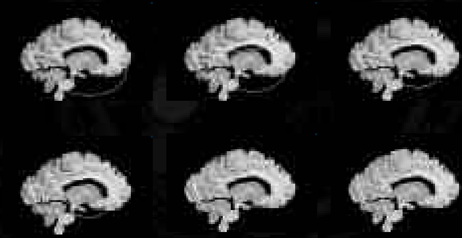
fast marching method

extrapolation across interface $I_t \pm \vec{N} \bullet \nabla I = 0$

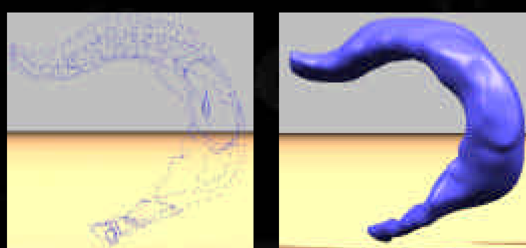
extrapolated variable

fast extension method

Image Segmentation - snakes



Interpolation of Unorganized Data Points "shrink wrap"



Coupled Particle - Level Set Method

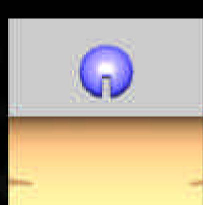
- level sets can have mass (volume) loss or gain, but give an aesthetically pleasing smooth surface representation

- particles maintain their mass, but do not generally form a smooth surface, especially when using a practical number of particles

- combining the two (in a clever way – based on the method of characteristics) gives very smooth surfaces without mass loss

Level Set Method

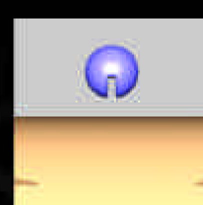
area loss due to regularization



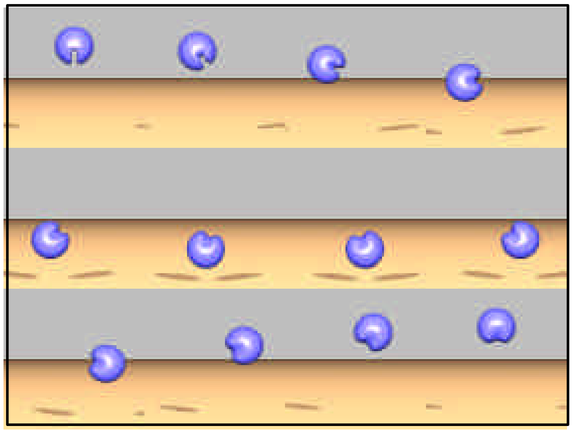
simple rigid body rotation

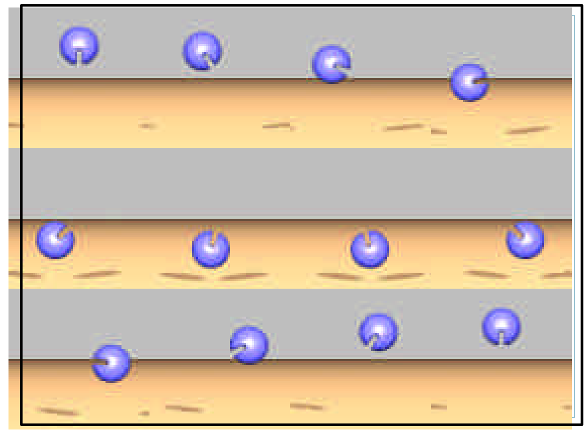
"New" Particle Level Set Method

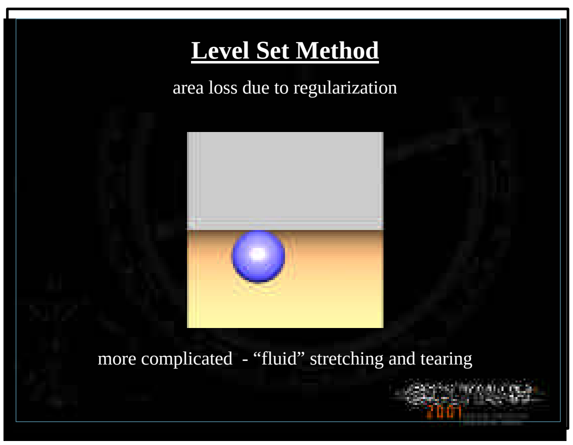
without area loss

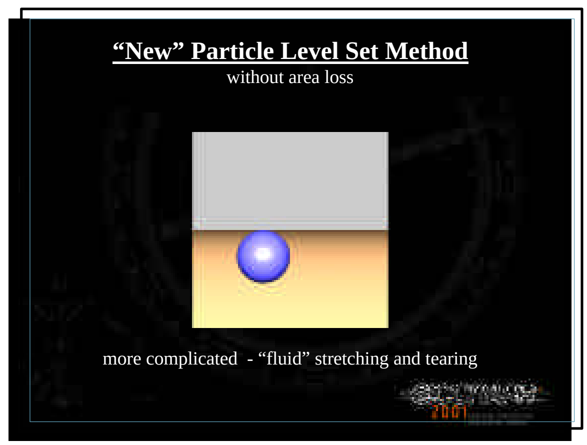


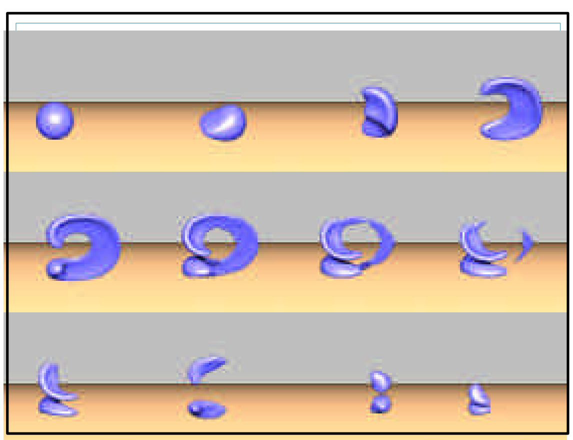
simple rigid body rotation

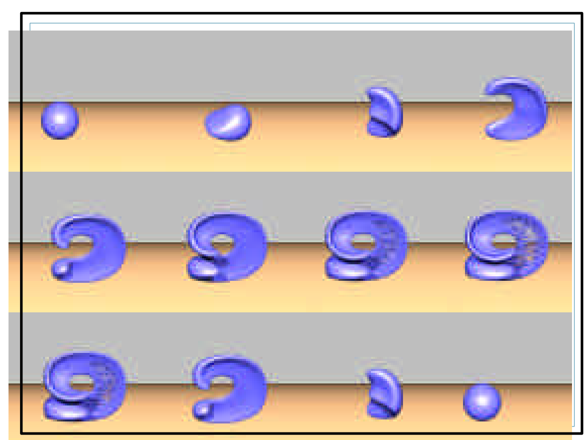





Level Set Method
area loss due to regularization
more complicated - “fluid” stretching and tearing


“New” Particle Level Set Method
without area loss
more complicated - “fluid” stretching and tearing





How to Finite Difference at Discontinuities?

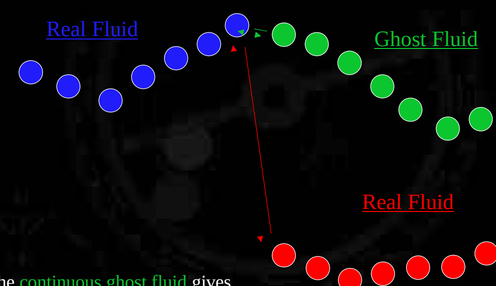
Differencing discontinuous quantities leads to O(1/Δx) terms that can produce large dissipation and dispersion errors.

One sided differencing ignores coupling mechanisms across the discontinuity.

Ghost Fluid Method

Define a set of "ghost" variables that are continuous across the interface. Then apply standard finite differencing both near and across the interface in a seamless fashion.

Ghost Fluid Method

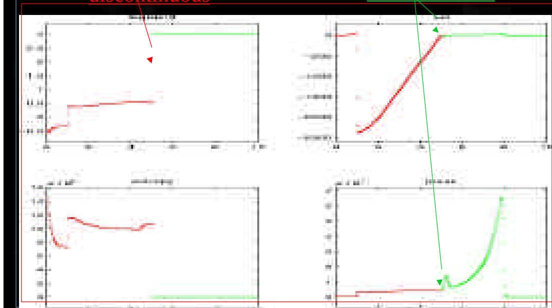


The continuous ghost fluid gives smaller numerical truncation errors than the discontinuous real fluid.

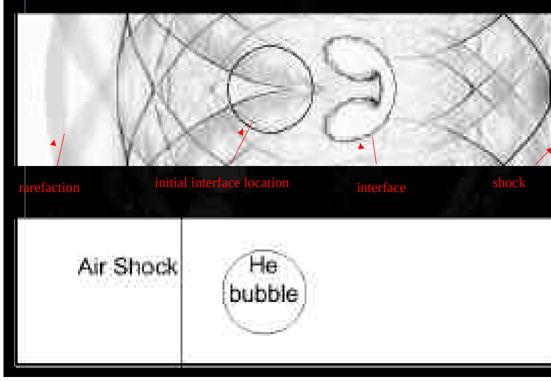
Air (RED) – Water (GREEN) Interface

Density is discontinuous

Pressure and Velocity are continuous



Helium-Air Interface



Variable Coefficient Poisson Equation $(b\,p_x)_x = f$

$[p]=s \qquad p_R = p_L + s$

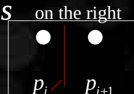
p discontinuous

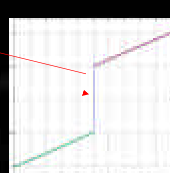
DEFINE $\hat{p} = p$ on the left

$\hat{p} = p - s$ on the right

$\hat{p}_R = \hat{p}_L$

$\hat{p}$ continuous





right hand side

$$(p_x)_{i+\frac{1}{2}} = \frac{\hat{p}_{i+1}-\hat{p}_i}{\Delta x} = \frac{(p_{i+1}-s)-p_i}{\Delta x} = \frac{p_{i+1}-p_i}{\Delta x} - \frac{s}{\Delta x}$$

standard symmetric matrix

Variable Coefficient Poisson Equation $(b\,p_x)_x = f$

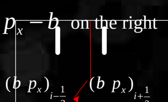
$[b\,p_x]=b \qquad (b\,p_x)_R = (b\,p_x)_L + b$

$b\,p_x$ discontinuous

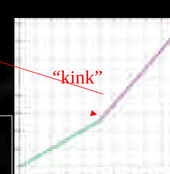
DEFINE $\widehat{b\,p_x} = b\,p_x$ on the left

$\widehat{b\,p_x} = b\,p_x - b$ on the right

$(\widehat{b\,p_x})_R = (\widehat{b\,p_x})_L$



$\widehat{b\,p_x}$ continuous



right hand side

$$((b\,p_x)_x)_i = \frac{(\widehat{b\,p_x})_{i+\frac{1}{2}} - (\widehat{b\,p_x})_{i-\frac{1}{2}}}{\Delta x} = \frac{(b\,p_x)_{i+\frac{1}{2}} - (b\,p_x)_{i-\frac{1}{2}}}{\Delta x} - \frac{b}{\Delta x}$$

standard symmetric matrix

Multidimensions $\nabla \bullet (b\,\nabla p) = f$

$[p] = s$ same as in 1D

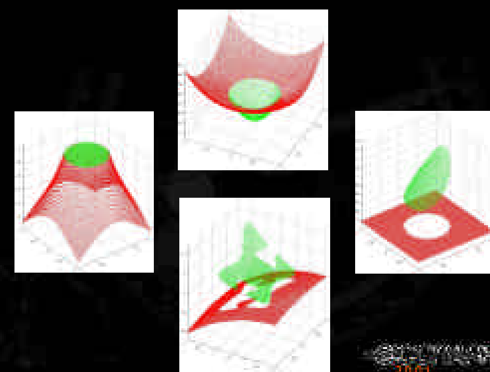
$[b\,\nabla p \bullet \vec{N}] = b$ dimension by dimension

$$[b\ p_x] = bn_1 \qquad [b\ p_y] = bn_2 \qquad [b\ p_z] = bn_3$$

$$[b\ p_x]n_1 + [b\ p_y]n_2 + [b\ p_z]n_3 = b$$

$$[b\,\nabla p \bullet \vec{N}] = b$$

arrows not valid in the reverse direction
smearing of the tangential derivative



Two-Phase Incompressible Flow

$$\vec{V}_t + \vec{V}\cdot\nabla\vec{V} + \frac{\nabla p}{r} = \frac{(\nabla\cdot t)^T}{r} + \vec{g} \qquad [r] \neq 0 \quad [m] \neq 0$$

Splitting…

$$\frac{\vec{V}^* - \vec{V}^n}{\Delta t} + \vec{V}\cdot\nabla\vec{V} = \frac{(\nabla\cdot t)^T}{r} + \vec{g}$$

and

Poisson Equation

$$\frac{\vec{V}^{n+1} - \vec{V}^*}{\Delta t} + \frac{\nabla p}{r} = 0 \quad \text{where} \quad \nabla \bullet \left(\frac{1}{r}\nabla p\right) = \frac{\nabla\cdot\vec{V}^*}{\Delta t}$$

Velocity Jump Conditions

$$[\vec{V}] = 0$$

$$[\nabla u\cdot\vec{T}_1] = [\nabla v\cdot\vec{T}_1] = [\nabla w\cdot\vec{T}_1] = 0$$

$$[\nabla u\cdot\vec{T}_2] = [\nabla v\cdot\vec{T}_2] = [\nabla w\cdot\vec{T}_2] = 0$$

$$[(\nabla u\cdot\vec{N}, \nabla v\cdot\vec{N}, \nabla w\cdot\vec{N})\cdot\vec{N}] = 0$$

$$[(\nabla u\cdot\vec{N}, \nabla v\cdot\vec{N}, \nabla w\cdot\vec{N})\cdot\vec{T}_1] = \cdots \neq 0$$

$$[(\nabla u\cdot\vec{N}, \nabla v\cdot\vec{N}, \nabla w\cdot\vec{N})\cdot\vec{T}_2] = \cdots \neq 0$$

$$\left[\frac{Du}{Dt}\right] = \left[\frac{Dv}{Dt}\right] = \left[\frac{Dw}{Dt}\right] = 0$$

Pressure Jump Conditions

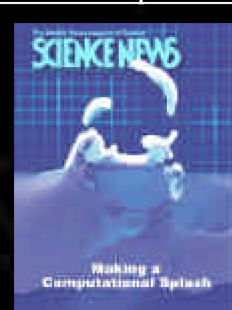
$$[p] = s\,k + 2[m](\nabla u\cdot\vec{N}, \nabla v\cdot\vec{N}, \nabla w\cdot\vec{N})\cdot\vec{N}$$

$$\left[\frac{p_x}{r}\right] = \left[\frac{(2mu_x)_x + (m(u_y + v_x))_y + (m(u_z + w_x))_z}{r}\right]$$

$$\left[\frac{p_y}{r}\right] = \left[\frac{(m(u_y + v_x))_x + (2mv_y)_y + (m(v_z + w_y))_z}{r}\right]$$

$$\left[\frac{p_z}{r}\right] = \left[\frac{(m(u_z + w_x))_x + (m(v_z + w_y))_y + (2mw_z)_z}{r}\right]$$

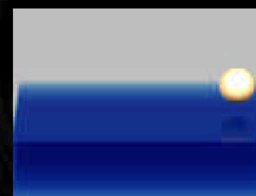
Two-Phase Incompressible Flow



Free Surface Flows

- incompressible Navier-Stokes equations on one side of the interface only (e.g. the water side)
- the fluid on the other side of the interface (e.g. the air side) has no dynamics – e.g. set $p = 1$ atmosphere
- Dirichlet pressure boundary conditions are applied at the interface, i.e. $p = 1$ atmosphere
- stress free boundary conditions are applied on the velocity field at the interface, i.e. the un-modeled fluid exerts no drag or resistance
- a 2nd order accurate symmetric method can be used to solve the Poisson equation with Dirichlet boundary conditions

Free Surface Flow
with object interaction



Control Particles for Spray Modeling

- in regions of high curvature, the particles are used to augment the level set function in order to alleviate mass loss

- in truly under-resolved regions (not enough grid points) the level set solution cannot be represented by the grid – even with the help of particles

- particles that "escape" the level set representation in under-resolved regions can be used as control particles for a spray modeling

Free Surface Flow
with object interaction

control particles are rendered here

Free Surface Flow
with object interaction

control particles are rendered here

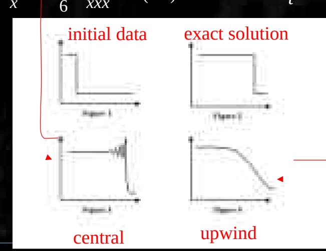
One Way Wave Equation $v_t + v_x = 0$

central $v_x \approx \frac{v_{i+1}-v_{i-1}}{2h}$

upwind $v_x \approx \frac{v_i - v_{i-1}}{h}$

$$v_t + v_x = -\frac{h^2}{6} v_{xxx} + O(h^4)$$

$$v_t + v_x = \frac{h}{2} v_{xx} + O(h^3)$$



Higher Order Approximation

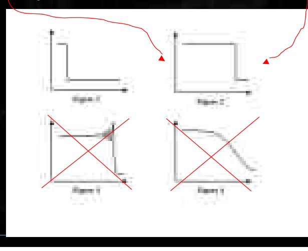
- add a new $h^r G$ term to cancel out the leading order error terms

central

$$v_t + v_x = \cancel{-\frac{h^2}{6} v_{xxx}} + h^r G + O(h^4)$$

upwind

$$v_t + v_x = \cancel{\frac{h}{2} v_{xx}} + h^r G + O(h^3)$$

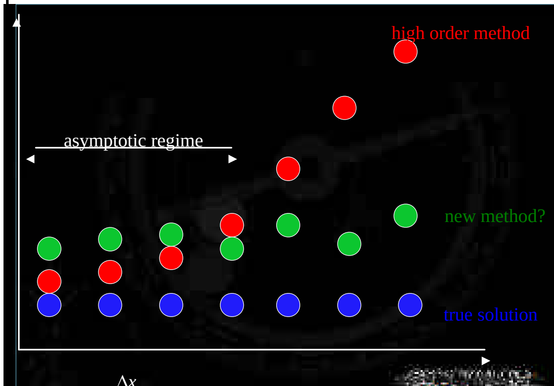


Key Points

- in traditional CFD, the results of numerical calculations are only meaningful when the computed solution is well-resolved
- well-resolved computations are within the convergent asymptotic regime where the numerical errors are proportional to the mesh spacing
- the only sensical $h^r G$ terms are those that accelerate convergence in the asymptotic regime, i.e. high order methods that cancel error

What happens in very complex flow fields where one cannot possibly use enough grid points to resolve all the important features of the flow field?

In general, one can claim very little about under resolved calculations on relatively coarse grids!

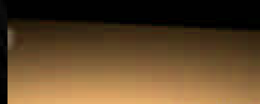


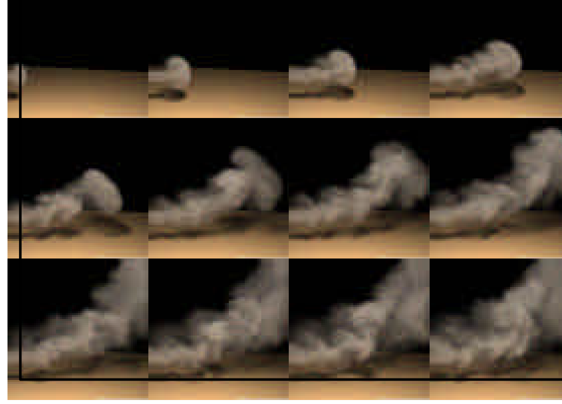
Vorticity Confinement – coarse grid fix

- vorticity $\omega = \nabla \times u$ needs help to overcome coarse grid dissipation
- locate the vorticity with $N = \frac{\nabla|\omega|}{|\nabla|\omega||}$
- calculate the magnitude and direction of the force that the vorticity should exert $N \times \omega$
- scale the force so that it vanishes for consistency, but still gives a good answer on a coarse grid $\epsilon \Delta x (N \times \omega)$

$$\boxed{u_t + u \cdot \nabla u + \frac{\nabla p}{\rho} = \mu \nabla^2 u + \epsilon \Delta x (N \times \omega)}$$

Smoke






Smoke